UDC 539.3

ON THE CONVERGENCE OF THE METHOD OF HOMOGENEOUS LINEAR APPROXIMATIONS IN PROBLEMS OF THE THEORY OF PLASTICITY OF INHOMOGENEOUS BODIES

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A formulation of the mathematical problem of equilibrium is given for a deformable solid whose elastic and plastic characteristics are functions of the coordinates. Extraction of the nonlinear terms and terms taking account of the deviation of the elastic moduli from some constant values results in the method of linear successive approximations, which is analogous to the method of elastic solutions. A problem of linear elasticity theory for a homogeneous solid subjected to fictitious mass and surface forces governed by the preceding approximation is solved in each approximation. Using the apparatus of functional analysis, the convergence of the method in application to the first and second boundary value problems is proved.

In many cases the solid and the medium are inhomogeneous relative to the mechanical properties. The inhomogeneity originates either during shaping of the material (solidification from the melt, tempering, ageing, etc.) or is the result of the presence of inhomogeneous temperature fields, racioactive exposure, and other physical and chemical fields. The elastic and plastic characteristics of the material are hence functions of the coordinates, which are introduced either explicitly (for example, for naturally inhomogeneous media), or by using field functions (the exposure dose [1], the degree of austenite transformation in tempering of steel [2], etc.).

1. Limiting ourselves to the class of isotropic materials, let us consider the elastic and plastic characteristics of a material to be functions of a number of parameters T, R, ..., dependent on the coordinates x_m $(m = 1, 2, 3)$ of the point. Let us assume that the functions $T(x_m)$, $R(x_m)$, ... can be determined independently of the determination of the stress and strain states of the solid (discrete problem), and that the process of the variation in the external loads, including T, R, ..., is such that a simple loading holds at each point of the solid. Then, for small elastic-plastic strains the typical problem of equilibrium of a solid is formulated as the following boundary value problem: It is necessary to determine 15 functions u_i, ε_{ij}, σ_{ij} satisfying the following relationships within the domain Ω occupied by the solid

$$\sigma_{ij,\,j} + F_i = 0, \qquad s_{ij} = \frac{2\sigma_u}{3\vartheta_u}\,\vartheta_{ij}$$

$$\sigma_u = \Phi(\vartheta_u, T, R), \quad \sigma = f(\varepsilon, T, R), \quad \varepsilon_{ij} = {}^1/_2\,(u_{i,\,j} + u_{j,\,i}) \qquad (1.1)$$

and on the boundary S of the domain Ω

$$u_i = \psi_i \quad \text{on} \quad S_u \qquad (S_u + S_\sigma = S)$$
$$\sigma_{ij} l_j = T_{\nu i} \quad \text{on} \quad S_\sigma \tag{1.2}$$

Here $\mathbf{u} = u_i \mathbf{x}_i$ is the displacement vector, $\mathbf{F} = F_i \mathbf{x}_i$ is the volume force vector, $\mathbf{T}_\nu = T_{i\nu}\mathbf{x}_i$ is the vector of the given surface forces, $\sigma_u = (^3/_2 s_{ij} s_{ij})^{1/2}$ is the stress intensity, $\sigma = {}^1/_3 \sigma_{ii}$ is the mean stress, $\partial_u = (^2/_3 \partial_{ij} \partial_{ij})^{1/2}$ is the intensity of small strains, $\varepsilon = {}^1/_3 \varepsilon_{ii}$ is the mean elongation, $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is the stress deviator, $\partial_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}$ is the strain deviator, T is the temperature, and R the exposure dose; summation over the twice-repeated Latin subscripts is kept in mind throughout. Without limiting the generality, only the two external parameters T and R are retained.

Let us represent the functions Φ and f as [1]

$$\sigma_u = 3G_0 \partial_u [1 - g(\partial_u, T, R)], \qquad \sigma = 3K_0(\varepsilon - \alpha T - qR)[1 + \varphi(T, R)]$$
$$g(\partial_u, T, R) = \frac{3G_0\partial_u - \Phi(\partial_u, T, R)}{3G_0\partial_u}, \qquad \varphi(T, R) = \frac{K(T, R) - K_0}{K_0} \tag{1.3}$$

where $G = G(T, R)$, $K = K(T, R)$ are the shear modulus and the volume modulus of elasticity of the material, G_0, K_0 are the same quantities for unexposed material at the temperature of the natural state T_0, and α, q are the coefficients of linear thermal and radiation expansion.

For the elastic state when $\sigma_u \leqslant \sigma_s(T, R)$, it must be assumed that

$$g(T, R) = \frac{G_0 - G(T, R)}{G_0}$$

where g and φ in this case are known functions of the coordinates given implicitly in terms of $T(x_m)$, $R(x_m)$.

The discrete problem of thermo-radiational plasticity under simple loading reduces to the determination of three functions $u_i(x_m)$ which satisfy the following equilibrium differential equations in Ω

$$(K_0 + {}^1/_3 G_0)\,\theta_{,i} + G_0\nabla^2 u_i + F_i - 3K_0(\alpha T + qR)_{,i} =$$
$$= 2G_0 [g(\partial_u, T, R)(\varepsilon_{ij} - \varepsilon\delta_{ij})]_{,j} - 3K_0[\varphi(T, R)(\varepsilon - \alpha T - qR)]_{,i} \tag{1.4}$$

and the boundary conditions

$$u_i = \psi_i \quad \text{on} \quad S_u$$
$$(\sigma_{ij}^{(e)} + \sigma_{ij}^{(T)})\, l_j = T_{\nu i} \quad \text{on} \quad S_\sigma \tag{1.5}$$

Here $\theta = \varepsilon_{ii}$ is the volume expansion, and we have used the notation

$$\sigma_{ij}^{(e)} = 3K_0\varepsilon\delta_{ij} + 2G_0(\varepsilon_{ij} - \varepsilon\delta_{ij}) - 3K_0(\alpha T + qR)\,\delta_{ij}$$
$$\sigma_{ij}^{(T)} = 3K_0\varphi(T, R)(\varepsilon - \alpha T - qR)\,\delta_{ij} + 2G_0 g(\partial_u, T, R)(\varepsilon_{ij} - \varepsilon\delta_{ij}) \tag{1.6}$$

Following the method of homogeneous linear approximations [1], we have the following formulation of the linear thermoelasticity problem for the $(n + 1)$-th approximation

$$(K_0 + {}^1/_3 G_0)\,\theta_{,i}^{(n+1)} + G_0\nabla^2 u_i^{(n+1)} + F_i - 3K_0(\alpha T + qR)_{,i} = \tag{1.7}$$
$$= 2G_0[g(\partial_u^{(n)}, T, R)(\varepsilon_{ij}^{(n)} - \varepsilon^{(n)}\delta_{ij})]_{,j} - 3K_0[\varphi(T, R)(\varepsilon^{(n)} - \alpha T - qR)]_{,i} \quad \text{in} \quad \Omega$$

under the boundary conditions

$$u_i^{(n+1)} = \psi_i \quad \text{on } S_u$$
$$\sigma_{ij}^{(e)\,(n+1)} l_j = T_{\nu i} - \sigma_{ij}^{(T)\,(n)} l_j \quad \text{on } S_\sigma \tag{1.8}$$

where all the quantities with superscript n are calculated by means of the values of $u_i^{(n)}$ from the n-th approximation.

The method of homogeneous linear approximations [1] is the following. In the zeroth approximation it is assumed that $g = \varphi = 0$, and (1.4), (1.5) reduce to the customary linear problem of thermoradiational elasticity for a homogeneous solid. Let its solution be $u_i^{(0)}(x_m)$. We evaluate $\varepsilon_{ij}^{(0)}$ and then $\vartheta_u^{(0)}$. If it turns out that $\vartheta_u^{(0)} \leqslant \varepsilon_s$ everywhere in Ω, where $\varepsilon_s = \varepsilon_s(T, R) = f_s(x_m)$ is the limit of the elastic strains at the considered point of the solid, then it must be assumed that $g = g(T, R)$ in (1.3), and we henceforth deal with the problem of thermoradiational elasticity for an inhomogeneous solid. Hence $g(T, R)$ and $\varphi(T, R)$ are given functions of the coordinates. In this case, by substituting the value $\varepsilon_{ij}^{(0)}$ in place of ε_{ij} in the right side of (1.4), and calculating $\sigma_{ij}^{(T)\,(0)}$ in conformity with (1.6) in terms of $\varepsilon_{ij}^{(0)}$, we arrive again at a linear problem of thermoradiational elasticity for a homogeneous solid in a first approximation. If $\vartheta_u^{(0)} > \varepsilon_s$ is some subdomain $\Omega_s^{(0)}$, then $g^{(0)} = g(\vartheta_u^{(0)}, T, R)$ in $\Omega_s^{(0)}$ and $g^{(0)} = g(T, R)$ outside $\Omega_s^{(0)}$. Using the values $\varepsilon_{ij}^{(0)}$ and $g^{(0)}$ to evaluate the right side of (1.4), and the values $\sigma_{ij}^{(T)}$ according to (1.6), we arrive at the problem of linear thermoradiational elasticity for a homogeneous solid with modified mass forces (or temperature) and surface loads.

2. Let us prove the convergence of the method of successive approximations for the first and second fundamental boundary value problems just as is done in [3] for isothermal plasticity problems.

Let us consider the space C_1 of vector functions $\mathbf{v}(x)$ defined in a volume $\bar{\Omega} = \Omega + S$, which are twice continuously differentiable and satisfy the condition $\mathbf{v} = 0$ on the surface S. In this space let it be given the scalar product and the norm by the formulas

$$(\mathbf{u}\cdot\mathbf{v})_{1\Omega} = \int_\Omega [(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})_0]\, d\Omega \tag{2.1}$$

$$\|\mathbf{v}\|_{1\Omega} = \sqrt{(\mathbf{v}\cdot\mathbf{v})_{1\Omega}} \tag{2.2}$$

Here the functions

$$(\mathbf{u}\cdot\mathbf{v}) \equiv \frac{3}{4}(u_{i,j} + u_{j,i})(v_{i,j} + v_{j,i}) - \theta_u\theta_v$$

$$(\mathbf{u}\cdot\mathbf{v})_0 \equiv \frac{3K_0}{2G_0}\theta_u\theta_v \qquad (\theta_u = u_{i,i},\ \theta_v = v_{i,i})$$

themselves satisfy all the axioms of the scalar product of two vector functions $\mathbf{u}$ and $\mathbf{u}$ at a point, except for the one axiom: $\mathbf{u} = 0$ does not follow from the conditions

$$\|\mathbf{u}\|_0 \equiv \sqrt{(\mathbf{u}\cdot\mathbf{u})} = 0, \qquad \|\mathbf{u}\|_0 \equiv \sqrt{(\mathbf{u}\cdot\mathbf{u})_0} = 0$$

Closing the space C_1 in the norm (2.2), we obtain some Hilbert space $H_{1\Omega}$.

Let us consider a space C_2 of vector functions $\mathbf{v}(x)$ $(x \equiv (x_1, x_2, x_3) \in \bar{\Omega})$ which are twice continuously differentiable and satisfy the conditions

$$\int_\Omega \mathbf{v}d\Omega = 0, \qquad \int_\Omega [\mathbf{r} \times \mathbf{v}]\, d\Omega = 0 \tag{2.3}$$

Let us introduce the scalar product $(\mathbf{u} \cdot \mathbf{v})_{2\Omega}$ and the norm $\| \mathbf{v} \|_{2\Omega}$ in C_2 by means of (2.1) and (2.2). Closing the space C_2 in the norm $\| \mathbf{v} \|_{2\Omega}$ we obtain some Hilbert space $H_{2\Omega}$. Considering (1.4) as an equation for the vector functions $\mathbf{u} = u_i \mathbf{x}_i$, let us multiply it scalarly by the vector function $\mathbf{v}$ and let us integrate over the volume. Then taking account of the appropriate boundary conditions for the first boundary value problem when $S_u = S$, $S_\sigma = 0$ in (1.5), we obtain

$$(\mathbf{u}\cdot\mathbf{v})_{1\Omega} = \frac{3}{2G_0}\int_\Omega F_i v_i d\Omega + \frac{9K_0}{2G_0}\int_\Omega [1 + \varphi(T, R)]\, (\alpha T + qR)\, \theta_v d\Omega +$$
$$+ \int_\Omega g(\partial_{ij}, T, R)\, (\mathbf{u}\cdot\mathbf{v})\, d\Omega - \int_\Omega \varphi(T, R)\, (\mathbf{u}\cdot\mathbf{v})_0\, d\Omega \tag{2.4}$$

It is assumed that the boundary condition of the first boundary value problem reduces to the homogeneous condition $\mathbf{u}|_s = 0$.

For the second boundary value problem when $S_u = 0$, $S_\sigma = S$ in (1.5), we obtain

$$(\mathbf{u}\cdot\mathbf{v})_{2\Omega} = \frac{3}{2G_0}\int_\Omega F_i v_i d\Omega + \frac{3}{2G_0}\int_S T_{\nu i} v_i dS +$$
$$+ \frac{9K_0}{2G_0}\int_\Omega [1 + \varphi(T, R)]\, (\alpha T + qR)\, \theta_v d\Omega + \int_\Omega g(\partial_u, T, R)\, (\mathbf{u}\cdot\mathbf{v})\, d\Omega -$$
$$- \int_\Omega \varphi(T, R)\, (\mathbf{u}\cdot\mathbf{v})_0\, d\Omega \tag{2.5}$$

Here and henceforth, integration in the Lebesgue sense is kept in mind.

We call the vector function $\mathbf{u} \in H_{1\Omega}$ $(\mathbf{u} \in H_{2\Omega})$ satisfying the integral relation (2.4) (the relation (2.5)) for any vector function $\mathbf{v} \in H_{1\Omega}$ $(\mathbf{v} \in H_{2\Omega})$ the generalized solution of the first (second) fundamental boundary value problem of thermoradiational plasticity. The generalized solutions of the problem satisfy the Lagrange principal of possible displacements; moreover, the generalized solution is classical if it is twice differentiable.

Let us define the operators A and B in the spaces $H_{1\Omega}$ and $H_{2\Omega}$ respectively, by the relationships

$$(A\mathbf{u}\cdot\mathbf{v})_{1\Omega} = \Pi_1(\mathbf{u}, \mathbf{v}) \qquad (\mathbf{v} \in H_{1\Omega}) \tag{2.6}$$

$$(B\mathbf{u}\cdot\mathbf{v})_{2\Omega} = \Pi_2(\mathbf{u}, \mathbf{v}) \qquad (\mathbf{v} \in H_{2\Omega}) \tag{2.7}$$

Here $\Pi_1(\mathbf{u}, \mathbf{v})$ and $\Pi_2(\mathbf{u}, \mathbf{v})$ are the right sides of relationships (2.4) and (2.5) respectively. If the governing right sides of these equations are linear, bounded functionals in the vector-functions $\mathbf{v}$, then according to the Riesz theorem, the operators A and B act in the spaces $H_{1\Omega}$ and $H_{2\Omega}$ respectively. In this case, seeking the generalized solution of the first boundary value problem reduces to the operator equation

$$A\mathbf{u} = \mathbf{u} \quad \text{in } H_{1\Omega} \tag{2.8}$$

and of the second boundary value problem, to the equation

$$B\mathbf{u} = \mathbf{u} \quad \text{in } H_{2\Omega} \tag{2.9}$$

It is natural to consider that the recurrent integral relations obtained from (1.7) in the same manner as (2.4), (2.5) from (1.4) will, taking account of the appropriate boundary conditions, determine the successive approximations of the generalized solutions of the appropriate fundamental boundary value problems. Taking account of (2.6), (2.7) we obtain that these recurrent relations for the first and second boundary value problems are, respectively

$$\mathbf{u}^{(n+1)} = A\mathbf{u}^{(n)} \quad \text{in } H_{1\Omega} \tag{2.10}$$

$$\mathbf{u}^{(n+1)} = B\mathbf{u}^{(n)} \quad \text{in } H_{2\Omega} \tag{2.11}$$

3. Let us prove the following two theorems.

Theorem 1. Let

1) $F_i(x) \in L_p(\Omega),\ p \geqslant {}^6/_5$;

2) $T(x),\ R(x) \in L_2(\Omega)$;

3) $g(\vartheta_u,\ T,\ R)$ and $\varphi(T,\ R)$ be Lebesgue measurable in Ω as functions of the variable $x = (x_1, x_2, x_3) \in \Omega$

4) for any $T,\ R,\ \vartheta_{u1}$ and ϑ_{u2} the following inequalities are valid:

$$\left| \frac{g(\vartheta_{u1}, T, R)\,\vartheta_{u1} \pm g(\vartheta_{u2}, T, R)\,\vartheta_{u2}}{\vartheta_{u1} \pm \vartheta_{u2}} \right| \leqslant \lambda < 1 \tag{3.1}$$

$$|\varphi(T,\ R)| \leqslant \beta < 1 \tag{3.2}$$

Then the operator A acts in the space $H_{1\Omega}$, the solution of (2.8) exists in $H_{1\Omega}$ and is the limit of the sequence (2.10) for any initial approximation $u^{(0)} \in H_{1\Omega}$.

Theorem 2. Let

1) all the conditions of Theorem 1 be satisfied;

2) $T_{\nu i}$ be summable on the boundary S with degree $r > {}^4/_3$; S satisfies the solvability conditions of the second boundary value problem of the theory of elasticity [4];

3) the principal vector and the principal moment of the system of external forces be zero.

Then the operator B acts in $H_{2\Omega}$, the solution of (2.9) exists in $H_{2\Omega}$ and is the limit of the sequence (2.11) for any initial approximation $u^{(0)} \in H_{2\Omega}$.

We start the proof of the theorems simultaneously. Let us note that the third condition of Theorem 2 is necessary since it is necessary for the existence of a solution to the second boundary value problem in the space of vector functions $\mathbf{v} \in H_{2\Omega}$ satisfying conditions (2.3), which exclude the displacement of a solid as an absolute solid.

It has been shown in [3, 5] that the first condition of Theorem 1 and the second condition of Theorem 2 yield, respectively, the boundedness of the functionals

$$\int_\Omega F_i v_i d\Omega \text{ in } H_{1\Omega} \text{ and } H_{2\Omega}, \quad \int_S T_{\nu i} v_i dS \text{ in } H_{2\Omega}.$$

The boundedness of the functional

$$\Phi_u(\mathbf{v}) \equiv \int_\Omega g(\vartheta_u,\ T,\ R)(\mathbf{u}\cdot\mathbf{v})\,d\Omega - \int_\Omega \varphi(T,\ R)(\mathbf{u}\cdot\mathbf{v})_0 d\Omega$$

in $H_{i\Omega}$ for any fixed $\mathbf{u} \in H_{i\Omega}$ $(i = 1, 2)$ follows from the measurability and boundedness of the functions g and φ.

Let us show the boundedness of the functional in the right side of (2.4) and (2.5) corresponding to the second and third members, in the spaces $H_{1\Omega}$ and $H_{2\Omega}$ We have

$$\left|\int_\Omega [1+\varphi(T,\ R)]\,(\alpha T+qR)\,\theta_v d\Omega\,\right| \leqslant (1+\beta)\,\left(\int_\Omega (\alpha T+qR)^2 d\Omega \int_\Omega \theta_v^2 d\Omega\right)^{1/2} \leqslant C\,\|\mathbf{v}\|_{i\Omega}$$

$$(C=\mathrm{const}>0,\ i=1,\ 2)$$

The existence of the integrals follows from the second and third conditions of Theorem 1. Therefore, the right sides of (2.6), (2.7) are bounded linear functionals in the vector functions $\mathbf{v}\in H_{i\Omega}$ $(i=1,2)$. Hence, it follows that the operators A and B act in the spaces $H_{1\Omega}$ and $H_{2\Omega}$ respectively.

For simplicity, we continue the proof for just Theorem 1. The proof of Theorem 2 is later carried out analogously.

Let us introduce the bilinear functionals

$$(\mathbf{u}\cdot\mathbf{v})_\Omega=\int_\Omega (\mathbf{u}\cdot\mathbf{v})\,d\Omega,\qquad (\mathbf{u}\cdot\mathbf{v})_{0\Omega}=\int_\Omega (\mathbf{u}\cdot\mathbf{v})_0 d\Omega \tag{3.3}$$

in the space $H_{1\Omega}$ which satisfy all the scalar product axioms except one: $\mathbf{u}=0$ does not follow in $H_{1\Omega}$ from the condition

$$\|\mathbf{u}\|_\Omega\equiv\sqrt{(\mathbf{u}\cdot\mathbf{u})_\Omega}=0\qquad\text{or}\quad \|\mathbf{u}\|_{0\Omega}\equiv\sqrt{(\mathbf{u}\cdot\mathbf{u})_{0\Omega}}=0$$

Evidently

$$(\mathbf{u}\cdot\mathbf{v})_{1\Omega}=(\mathbf{u}\cdot\mathbf{v})_\Omega+(\mathbf{u}\cdot\mathbf{v})_{0\Omega},\qquad \|\mathbf{u}\|_{1\Omega}=\sqrt{\|\mathbf{u}\|_\Omega^2+\|\mathbf{u}\|_{0\Omega}^2} \tag{3.4}$$

Let us consider the subset $H_{1\Omega}^{\mathrm{int}}$ of the space $H_{1\Omega}$, whose elements $\mathbf{u}$ satisfy the condition $\|\mathbf{u}\|_{0\Omega}=0$, i.e.,

$$H_{1\Omega}^{\mathrm{int}}=\{\mathbf{u}\in H_{1\Omega}:\|\mathbf{u}\|_{0\Omega}=0\} \tag{3.5}$$

This latter condition means that $\theta_u=0$ almost everywhere in Ω. It can be shown that $H_{1\Omega}^{\mathrm{int}}$ is a complete subspace of $H_{1\Omega}$, i.e., a closed linear manifold of the space $H_{1\Omega}$, where the first bilinear functional in (3.3) satisfies all the scalar product axioms in $H_{1\Omega}^{\mathrm{int}}$ and agrees with the scalar product (2.1) introduced in all of $H_{1\Omega}$.

It can be shown analogously that the subset

$$H_{1\Omega}^{\mathrm{div}}=\{\mathbf{u}\in H_{1\Omega}:\|\mathbf{u}\|_\Omega=0\} \tag{3.6}$$

of the space $H_{1\Omega}$ is also a complete subspace and the second bilinear functional in (3.3) is a scalar product therein, which agrees with the scalar product (2.1) of the space $H_{1\Omega}$.

It is seen that the subspaces $H_{1\Omega}^{\mathrm{int}}$ and $H_{1\Omega}^{\mathrm{div}}$ are orthogonal and are maximal subspaces in which the functions (3.3) are, respectively, the scalar products. It can hence be shown that the direct sum of $H_{1\Omega}^{\mathrm{int}}$ and $H_{1\Omega}^{\mathrm{div}}$ yields the whole space $H_{1\Omega}$, i.e., any vector function $\mathbf{u}\in H_{1\Omega}$ is uniquely representable as

$$\mathbf{u}=\mathbf{u}'+\mathbf{u}''\qquad(\mathbf{u}'\in H_{1\Omega}^{\mathrm{int}},\ \mathbf{u}''\in H_{1\Omega}^{\mathrm{div}}) \tag{3.7}$$

Taking account of the above, (2.4) can be represented as

$$(\mathbf{u}'\cdot\mathbf{v}')_\Omega+(\mathbf{u}''\cdot\mathbf{v}'')_{0\Omega}=\frac{3}{2G_0}\int_\Omega F_i v_i d\Omega+\int_\Omega g\,(\partial_u,\ T,\ R)\,(\mathbf{u}'\cdot\mathbf{v}')\,d\Omega-\int_\Omega \varphi\,(T,R)(\mathbf{u}''\cdot\mathbf{v}'')_0 d\Omega+$$

$$+\frac{9K_0}{2G_0}\int_\Omega [1+\varphi\,(T,\ R)]\,(\alpha T+qR)\ \theta_{v''}d\Omega \tag{3.8}$$

The relationship (3.8) for the desired $\mathbf{u}=\mathbf{u}'+\mathbf{u}''$ should hold for any $\mathbf{v}=\mathbf{v}'+\mathbf{v}''$, i.e., for any pair $\mathbf{v}'\in H_{1\Omega}^{\mathrm{int}}$ and $\mathbf{v}''\in H_{1\Omega}^{\mathrm{div}}$. Setting $\mathbf{v}''=0$, we obtain

$$(\mathbf{u}'\cdot\mathbf{v}')_\Omega = \int_\Omega g(\vartheta_u, T, R)(\mathbf{u}'\cdot\mathbf{v}')\,d\Omega + \frac{3}{2G_0}\int_\Omega F_i v'_i\,d\Omega \tag{3.9}$$

for any v'. Assuming $v' = 0$, we obtain for any v''

$$(\mathbf{u}''\cdot\mathbf{v}'')_{0\Omega} = -\int_\Omega \varphi(T, R)(\mathbf{u}''\cdot\mathbf{v}'')_0\,d\Omega + \frac{3}{2G_0}\int_\Omega F_i v_i''\,d\Omega + \\ + \frac{9K_0}{2G_0}\int_\Omega [1+\varphi(T, R)](\alpha T + qR)\,\theta_{v''}\,d\Omega \tag{3.10}$$

It is seen that from (2.4) we have gone over to an equivalent pair of relationships (3.9), (3.10), i.e., the generalized solution of the first boundary value problem of thermoradiational plasticity exists if and only if each of the equations (3.9), (3.10) has a solution in the appropriate subspace $H_{1\Omega}^{\text{int}}$ or $H_{1\Omega}^{\text{div}}$.

Analogously, we can pass from (2.6) over to a pair of relationships for any $\mathbf{v}'$ and any $\mathbf{v}''$ governing the operators A' and A'' in the subspaces $H_{1\Omega}^{\text{int}}$ and $H_{1\Omega}^{\text{div}}$, respectively

$$(A'\mathbf{u}'\cdot\mathbf{v}')_\Omega = \Pi_1'(\mathbf{u}', \mathbf{v}') \tag{3.11}$$

$$(A''\mathbf{u}''\cdot\mathbf{v}'')_{0\Omega} = \Pi_1''(\mathbf{u}'', \mathbf{v}'') \tag{3.12}$$

Here $\Pi_1'(\mathbf{u}', \mathbf{v}')$ and $\Pi_1''(\mathbf{u}'', \mathbf{v}'')$ are the right sides of (3.9) and (3.10), respectively. The operator A evidently acts in the space $H_{1\Omega}$ if and only if the operators A' and A'' act in the subspaces $H_{1\Omega}^{\text{int}}$ and $H_{1\Omega}^{\text{div}}$, where $A'\mathbf{u}' + A''\mathbf{u}'' = A\mathbf{u}$. In this case, (2.8) in $H_{1\Omega}$ is equivalent to the pair of equations

$$A'\mathbf{u}' = \mathbf{u}' \quad \text{in } H_{1\Omega}^{\text{int}} \tag{3.13}$$

$$A''\mathbf{u}'' = \mathbf{u}'' \quad \text{in } H_{1\Omega}^{\text{div}} \tag{3.14}$$

and the convergence of the sequence (2.10) in $H_{1\Omega}$ is equivalent to the pair of convergences of the sequences

$$\mathbf{u}'^{(n+1)} = A'\mathbf{u}'^{(n)} \text{ in } H_{1\Omega}^{\text{int}} \tag{3.15}$$

$$\mathbf{u}''^{(n+1)} = A''\mathbf{u}''^{(n)} \text{ in } H_{1\Omega}^{\text{div}} \tag{3.16}$$

Hence, to prove the theorem it is sufficient to show the convergence of each of the sequences (3.15), (3.16) in the appropriate subspace. It is seen that (3.13) expresses the problem of thermoradiational plasticity for an incompressible material, and (3.15) determines the sequence of approximate generalized solutions of this problem. The convergence of the Il'iushin method of elastic solutions is proved in [5] for incompressible materials in the case of using the so-called reduced shear modulus. In the case of the thermoradiational plasticity problem for an incompressible material, the convergence of the sequence (3.15) to the solution of (3.13) in the subspace $H_{1\Omega}^{\text{int}}$ can be proved analogously by using the condition (3.1) imposed on the function $g(\vartheta_u, T, R)$.

The convergence of the sequence (3.16) to the solution of (3.14) follows from the compressive property of the operator A'' in the subspace $H_{1\Omega}^{\text{div}}$, i.e., from the relationship

$$\| A''\mathbf{u}_1'' - A''\mathbf{u}_2'' \|_{0\Omega} \leqslant \beta \| \mathbf{u}_1'' - \mathbf{u}_2'' \|_{0\Omega} \tag{3.17}$$

for any $\mathbf{u}_1'', \mathbf{u}_2'' \in H_{1\Omega}^{\text{div}}$. Let us prove this property by using the constraint (3.2) imposed on the function $\varphi(T, R)$.

From (3.12) we have

$$((A''\mathbf{u}_1'' - A\mathbf{u}_2'')\cdot\mathbf{v}'')_{0\Omega} \leqslant \left|\int\limits_{\Omega} \varphi(T, R)\,((\mathbf{u}_1'' - \mathbf{u}_2'')\cdot\mathbf{v}'')_0\, d\Omega\right| \leqslant$$
$$\leqslant \beta \int\limits_{\Omega} \|\mathbf{u}_1'' - \mathbf{u}_2''\|_0 \cdot \|\mathbf{v}''\|_0\, d\Omega \leqslant \beta\, \|\mathbf{u}_1'' - \mathbf{u}_2''\|_{0\Omega} \cdot \|\mathbf{v}''\|_{0\Omega}$$

Setting $\mathbf{v}'' = A''\mathbf{u}_1'' - A''\mathbf{u}_2''$, we arrive at (3.17). Theorem 1, and by analogy, Theorem 2 also have been proved.

Thus, the convergence of the method of homogeneous linear approximations (the method of homogeneously elastic solutions) has been proved under the conditions of these theorems in application to the first and second fundamental boundary value problems of thermoradiational elasticity and plasticity.

4. Let us examine the constraints (3.1) and (3.2) imposed on the change in the elastic or plastic properties of the material during radioactive exposure and heating (cooling). The constraint on the function $g\ (\Theta_u,\ T,\ R)$ holds under the conditions

$$3G_{\text{sup}} \geqslant \frac{\sigma_u(\Theta_{u1}, T, R) \pm \sigma_u(\Theta_{u2}, T, R)}{\Theta_{u1} \pm \Theta_{u2}} \geqslant a > 0$$
$$\sigma_u(0, T, R) = 0, \quad a = \text{const}, \quad G_{\text{sup}} = \text{const}, \quad G_{\text{sup}} < 2G_0 \qquad (4.1)$$
$$\left(\lambda = \max\left\{\left|1 - \frac{G_{\text{sup}}}{G_0}\right|,\ \left|1 - \frac{a}{3G_0}\right|\right\}\right)$$

These conditions mean that the curve of the dependence $\sigma_u \sim \Theta_u$ at any point of the solid starts from the origin and lies, together with the directional vector of all its tangents, within an acute angle formed by two half-lines starting from the origin and the tangents $3G_{\text{sup}}$ and a of the slopes to the horizontal axis. The conditions (4.1) are satisfied in a broad range of variation of T and R for an extensive class of plastic and elastic-plastic materials. They are valid, say, for hardening elastic-plastic solids with a convex $\sigma_u \sim \Theta_u$ curve if $G_{\text{sup}} \equiv \sup_\Omega\ G\ (T,\ R) < 2G_0$.

In the particular case of the thermoradiational elasticity problem when $g\ (\Theta_u,\ T,\ R) \equiv g(T,R)$, condition (3.1) is equivalent to the condition $|\ g\ (T,\ R)\ | \leqslant \lambda < 1$, which means that the shear modulus in a solid under radioactive exposure and heating (cooling) should not change by more than $\lambda \cdot 100\%$.

The constraint (3.2) imposed on the function $\varphi\ (T,\ R)$ means that the relative change in the modulus of multilateral compression in a solid under exposure and heating should not exceed the constant $\beta < 1$ in absolute value.

Conditions (3.1), (3.2) are only sufficient for the convergence of the method of homogeneously linear solutions. It can be shown that these conditions can be weakened in some particular examples.

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